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Calculating Transformities with an Eigenvector Method

Dennis Collins and Howard T. Odum

ABSTRACT

Eigenvalues-eigenvectors were used to calculate transformities from sets of emergy equations representing energy transformations. Transformity = emergy, spelled with an "m," divided by available energy. This procedure extends the calculation of energy quality measures introduced by Murray Patterson in 1983. The method uses energy systems diagrams to organize data on the flow of available energy through transformation processes. For each transformation process, an equation is written with input emergy equal to that of an output. After the data are combined with a matrix equation, transformities are calculated by minimizing eigenvalues. A program is provided for the commercial software MATHEMATICA, which solves the simultaneous emergy equations and prints out a vector of transformities. Comparisons are made between the transformities determined in this way with those previously estimated with other methods. Examples include simple configurations, the ecosystem in Silver Springs, Florida, and the estuary in Louisiana used by Tennenbaum (1988).

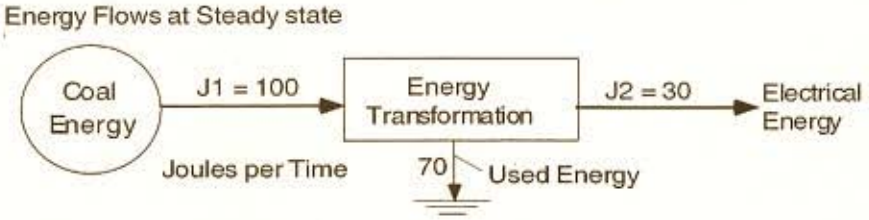
INTRODUCTION

Emergy, spelled with an "m" is the available energy (exergy) of one kind required to generate the available energy (exergy) of other kinds. Calculating emergy has theoretical and practical importance to energy analysis and public policies based on evaluating work of nature and the economy on a common basis. This paper provides a convenient method for calculating transformities (the emergy per unit available energy) from energy data of the environment and economy.

BACKGROUND

Energy systems networks can be used to represent the essence of real systems, including complex energy flows and transformation processes. The systems are often made quantitative by estimating values of energy flows into and out of each transformation. Numerical values can be written on the pathways of systems diagrams. Figures 1-6 give examples of such energy sysems networks in which the symbols carry additional mathematic and energetic meaning (Odum, 1971, 1983, 1996). The diagrams represent the energy transformation hierarchy concept by the position of items and flows. Flow of available energy decreases as the quality of energy increases with successive energy transformation processes from left to right. Each joule is capable of doing more when interacting with other energy flows. The energy flows can also be represented in tabular ways amenable to matrix mathematics.

This concept of embodied energy was developed in the 1960's and given the name emergy in 1983 (Odum, 1986; Scienceman, 1987) so that items of all kinds (environment, fuels, chemicals, information) could be put on a common basis by expressing energy flows in units of one kind of available energy previously used and expressed as emjoules. The emergy for one unit of available energy was defined as the transformity. Transformities measure the position of each kind of energy in the natural energy hierarchy in which different kinds are on different scale in the universal series of energy transformations. Available energy (exergy) of different kinds should not be added to imply work until



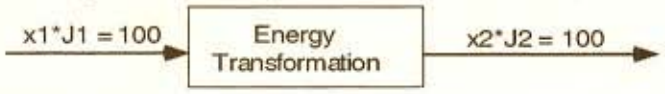
Emergy is the available energy of one kind used up to generate a flow
Emjoule is the unit of emergy
Transformity = emergy per unit energy with units: emjoules/joule

In this example::
Emergy flow of both input and outputs = 100 coal emjoules/time

x1 = 1 by definition (the coal transformity of coal)

x2 = Coal transformity of electricity = (100 coal emjoules/time)/(30 electrical joules/time)
= 3.33 coal emjoules/joule electricity

Balance of emergy flows at steady state, 100 coal emjoules per time



Emergy Flow Equation: $x1*J1 = x2*J2$

$x1*J1 - x2*J2 = 0$

Figure 1. *An energy transformation with definitions.*

multiplied by transformities to represent each in emergy units of one kind (Odum, 1996). Most transformation processes have more than one kind of energy input, which can be put on a common basis as the emergy of one kind.

Where a system has been aggregated so that outputs are fed back as necessary interactions, and for a process at steady state with storages constant, the emergy inputs to the transformation process may be conserved in the output products. An emergy balance equation can be written for inputs and outputs with emergy out equal to the sum of the emergy flows in (Figure 1). Emergy of any flow is the product of the energy times the transformity relating the emergy for one kind to that type of energy. Transformity was given various names during the development period and renamed transformity in 1983 (Odum, 1976, 1986, 1987). After 1983, to avoid fractions, solar emergy was used, expressing transformities in terms of the solar energy, the energy type with the largest flows but lowest concentration. In a summary book, ten ways of estimating transformities were given with examples (Odum 1996).

In 1983 and later papers, Murray Patterson (1983, 1984, 1993, 1998) expressed energy transformation equations of systems networks in a matrix in which each row is a transformation process equation and each energy flow is placed in a column according to the transformity. Figure 2 is a two process example in which x's are transformities. Then he used linear algebra methods to estimate the coefficients from the properties of the set of equations for the whole network, adjusting the values to minimize the residual with a least squares method. The coefficient of one type of energy was given the value 1, and the mathematical solution generated the other coefficient values. Patterson's method of energy systems analysis appears to be an important way of estimating emergy and transformity. This

paper provides an eigenvalue method and a convenient computer program to facilitate these calculations with standard software.

As Patterson has explained, real energy transformation networks evaluated for study may not be in steady state. They may be incomplete in representing main energy flows. They may not be aggregated so that each output carries all the input emergy. The calculations can still be made, and the results used to describe the energy network, but the coefficients obtained in these cases will have additional error in representing transformity that is defined as the necessary emergy required to relate energy on one scale to that on another.

Emergy Equations and Definitions

Figure 1 illustrates definitions with one energy transformation. Coal exergy is converted by a transformation to electrical exergy. Emergy is defined as the available energy of one kind previously used up in a transformation to make a useful energy flow of a different kind. Therefore, coal emergy in coal emjoules required to generate one joule of electrical energy is called the coal transformity. It is the quotient of one type of emergy flow divided by the energy flow of another kind. The example in Figure 1 was simplified to contain only one input and one output. J's are energy flows, and x's are transformities of Type 1 energy.

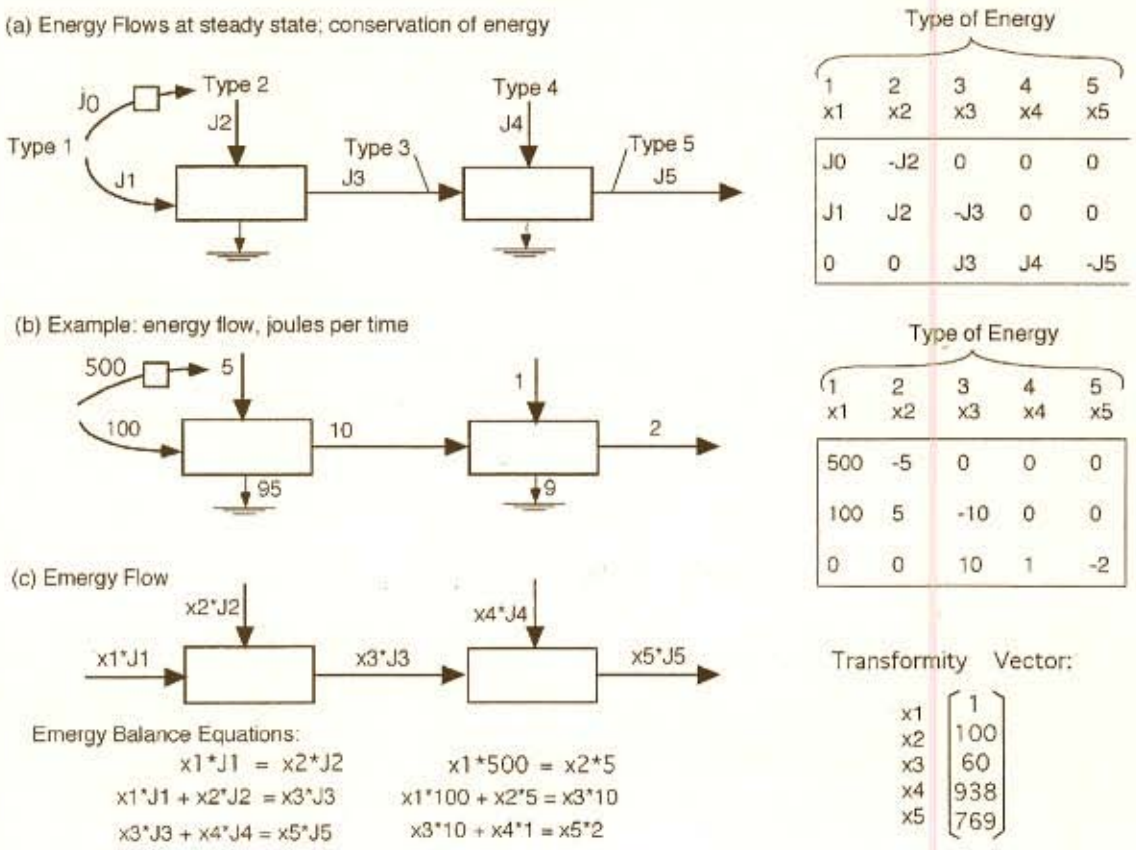


Figure 2. *Energy flows, emergy flows, and emergy transformation equations for a series of two energy transformations. (a) Energy flows (J's); (b) numerical example; (c) emergy flows and equations; tabular form for representing energy flows in a matrix used by computer programs to calculate transformities (x's).*

In a real energy systems network, there are usually two or more inputs to each transformation (Figure 2). In any energy transformation, such as those in Figure 2a, the available energy used and the energy output from each transformation can be expressed in the same emergy units equal to the energy flow (J's) multiplied by one kind of transformity (x's). Pathways of exergy dissipation have zero emergy flow and are omitted. If the lowest transformity energy is solar, the units are solar emjoules. Figure 2c has equations with an expression for the solar emergy flow accompanying each energy flow. The J's are the flows of energy resulting from the previous transformations. The x's are the solar transformities.

There are as many such emergy equations in an energy network as there are energy transformations. If the terms of each equation in Figure 2 are gathered on one side of the equals sign, the output flows (on the right) become negative terms:

$$x1*J1 + x2*J4 - x3*J3 = 0 \quad (1)$$

$$x3*J3 + x4*J4 - x5*J5 = 0$$

The equations can also be shown in a tabular form with the transformities (x's) across the top (Figure 2). Energy flow data in this form are expressed in matrix form as

$$Mx = 0 \quad (2)$$

where M is the matrix of energy flow and x is the vector of transformities.

Drawing the diagrams and placing the energy flow values on the pathways is a way of making sure energy flows fit the following energy laws. (1) The energy outflows at steady state equal the inflows. (2) Available energy flows decrease in passing through a transformation. (3) If the diagram is drawn with the left-right convention, the highest transformity with the smallest energy flows enters from the right as a control to the transformation. The lowest transformity with the most energy inflows from the left. The transformed output is intermediate in available energy content and transformity.

EIGENVALUE-EIGENVECTOR METHOD FOR CALCULATING

Emergy and Transformity

In papers starting in 1983, Patterson (1983, 1984, 1993) worked the web evaluating procedure backwards, deriving the transformities from the set of equations where only the energy flows (J's) were available. One type of energy was given a transformity of 1 (for electricity in some examples). He used a least squares method to find the transformities that solve equation 2 with the least error. The following is the new procedure using eigenvalue-eigenvectors.

In the matrix equation: $Mx = 0$, M is an m x n matrix of m processes and n ingredient inputs and outputs. Matrix M has the energy flows arranged in columns according to energy type as in Figures 2-4. The vector x is the a column vector of coefficients (transformities) for these types of energy. An eigenvector has the same result as the matrix when the eigenvector is multiplied by a set of numbers that are then called eigenvalues (Appendix). The matrix product of Mx is set equal to Lx where the L is the matrix of eigenvalues for an eigenvector x made up of the transformities.

$$Mx = Lx \quad (3)$$

By finding an eigenvector which has almost zero eigenvalues, the matrix equation approaches:

$$Mx = 0 \quad (4)$$

where the values of the vector x is the set of transformities. In other words, the procedure solves the equation. Whatever small values there are for the minimum eigenvalues L of that vector are the residue. The residue exists because the equations and data are not perfect. For analyses of real systems where data and equations are not perfect representations of the real transformations, the residual (error) can be represented with the vector e in equation (5)

$$Mx + e = 0 \qquad (5)$$

The Appendix from Collins (1998) explains the mathematics further.

EVALUATING TRANSFORMITIES WITH MATHEMATICA

The program for the commercial software program MATHEMATICA is included (Table 1) so anyone can rapidly evaluate emergy and transformities from data of network energy flows with the following procedure:

1. Load the program MATHEMATICA. To see if everything is ready, type in 2+2 and press the SHIFT Key and ENTER. If things are working OK, the program calculates and places the answer 4 on the screen.
2. Either type in the program in Table 1 or paste it to the MATHEMATICA screen via the clipboard, from a stored file (available from the authors).
3. Type in the emergy equations like the ones in equation #1 above. At the top of the blank screen write the matrix of processes, one row for each energy transformation equation. In each equation one or more ingredient input terms of the process are written with a plus sign and one output term with a minus sign. The first number requires .0 to be included. The following is an example which shows the use of punctuation without spaces for the two row, five column data matrix in Figure 2:

```
m= {{100.0, 5,-10,0,0},                    (6)
    {0, 0, 10,1,-2}};
```

4. Press the SHIFT KEY and ENTER. The software follows the instructions in the program (Table 1) and prints out the transformity vector x. The items of the matrix of energy data are in the same order as the column headings (kinds of energy). The program divides through by the smallest transformity and rounds to get whole numbers.

Table 1. Program to enter into MATHEMATICA in order to calculate transformities from emergy equations

```
m = {                }
MatrixForm[m]
a = Transpose[m] .m;
p = Eigenvectors[a];
err = m.Transpose[p];
MatrixForm[err];
MatrixForm[Transpose[p]];
Eigenvalues[a];
u = Min[Abs[Take[Eigenvectors[a], -1]]];
t = (1/u) Take[Eigenvectors[a], -1];
MatrixForm[Transpose[t]]
```

m = matrix of energy flows from the emergy equations.

5. Save the result as a file and/or print out the screen with the PRINT item in the FILE menu. Label the transformities with the name of the energy type.

Table 2 is an example of the result of running the program with the matrix for the simple example in Figure 3 set up to be in steady state with emergy in equal emergy out. The matrix that was entered was printed out so you can check it. At the bottom the transformities for the three kinds of energy were printed as a column vector: 1, 1000, and 100. For this simple case each transformity can be checked by dividing input emergy by output energy.

Explanation of the Program

Each line in the program (Table 1) is a step in the calculations. Where the lines end in a semicolon, the result of the calculation is not printed out. Remove the semicolon from any line which you want to print. As written in Table 1, only the input matrix, minimum eigenvalues of the eigenvector indicating residue, and the transformity vector are printed.

The program first instructs the software to find the matrix $a = M^T M$ where M^T is the transpose of M.

Next it finds the eigenvalues and corresponding eigenvectors of the matrix a. In MATHEMATICA the eigenvectors are the columns of the transpose of the matrix p = eigenvectors[a].

The program finds the residual (error) matrix err = M*Transpose[p] (in MATHEMATICA). A given column represents the error of each process (row of M) according to the given column valuation. The most efficient processes are those with the most negative error, supposing the components of the corresponding eigenvector are all positive. More explanation is given in Appendix A.

Table 2. Output of MATHEMATICA evaluating the three transformation system in Figure 3

```
m = {{100.0, 1, -11},
   {100, 2, -21},
   {100, 0, -1}};

MatrixForm[m]
a = Transpose[m].m;
p = Eigenvectors[a];
err = m.Transpose[p];
    MatrixForm[err];
MatrixForm[Transpose[p]];
Eigenvalues[a]
u = Min[Abs[Take[Eigenvectors[a], -1]]];
t = (1/u) Take[Eigenvectors[a], -1];
MatrixForm[Transpose[t]]
```

$$\begin{pmatrix} 100. & 1 & -11 \\ 100 & 2 & -21 \\ 100 & 0 & -1 \end{pmatrix}$$

$$\{30368.5,\ 199.549,\ 7.41222 \times 10^{-15}\}$$

$$\begin{pmatrix} 1. \\ 1000. \\ 100. \end{pmatrix}$$

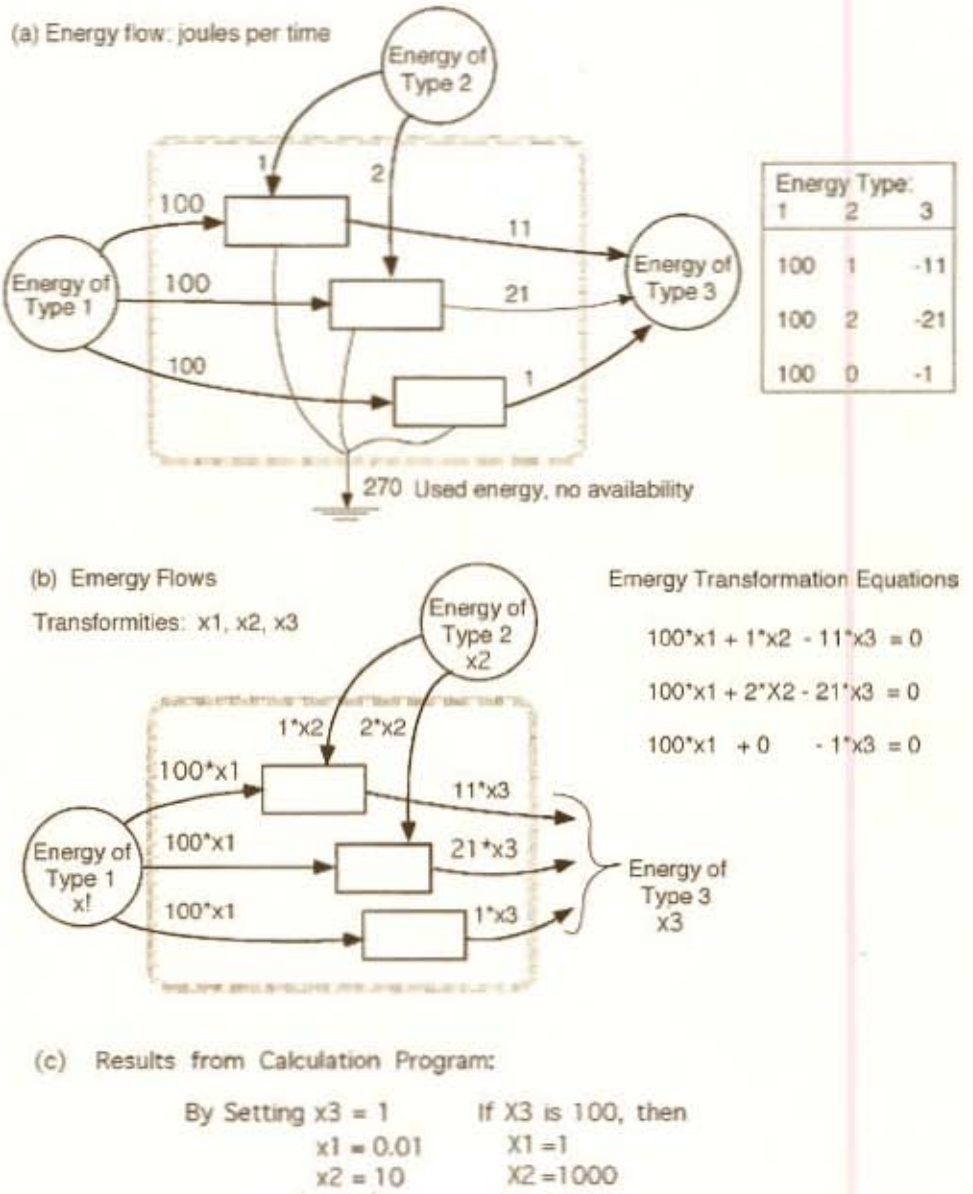


Figure 3. *An energy web with three energy transformations and its emergy equations. (a) Energy flow and matrix for entering data; (b) emergy equations and flows.*

At the end the program divides the eigenvector (with the minimum eigenvalues) by the smallest emergy, which becomes unity and the rest become whole numbers (transformities in emjoules of the lowest energy type in the group). Thus, the final vector is expressed in transformities of the lowest quality type of energy. These can be converted into solar transformities in solar emjoules/joule by multiplying them each by the solar transformity of the lowest energy type in the group.

EVALUATION EXAMPLES

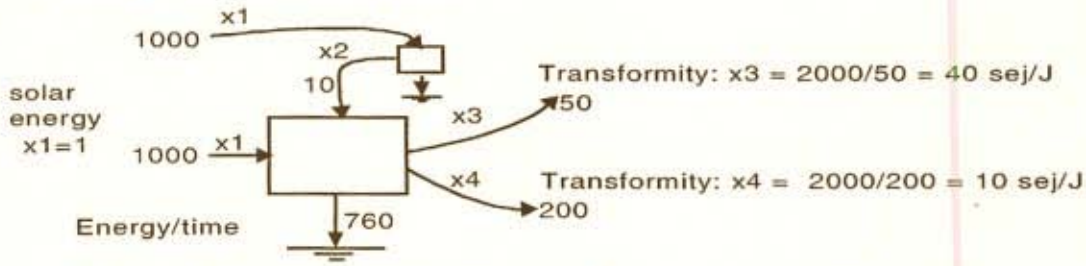
Evaluating Transformations with Energy Flows that Branch

Whereas the systems in Figures 2 and 3 had one output of each transformation without branching, most networks have energy flows that branch. In aggregating the real world detail in an energy systems diagram, two kinds of branches may be arranged. In the matrix a separate line is required for each branch.

A co-product branch has two outputs of different energy type and transformity based on the same input emergy. Figure 4 has an example with two input energy flows. For the transformation co-products for one line of the matrix, enter inputs of 100 and 10 and an output of -50. For the other line, enter the same inputs 100 and 10 and a different output -200. For this simple example, you can check the output transformities by dividing each output energy into the total input solar emergy = 2000 sej (Figure 4a).

A split divides an energy flow of one kind into two flows, both of the same energy type and transformity. The emergy is divided in the same proportion. A little energy disperses in the process. For example, in Figure 5, the inputs to the organics are 2.8 and 57 with -40 the product. The energy of organics splits 40 into flows of 15 and 25. All three have the same transformity and can be entered in the same column. One branch of the split has the input 15 and output -13.8; the other branch has input 25 and output -24.2.

(a) Block with Co-products, Outputs with Different Transformities
Energy Flows at Steady State



(b) Emergy Equations for Eigenvector Program, one output each:

```
x1*100 = x2*10
x1*1000  + x2*10 = x3*50
x1*1000  + x2*10 = x4*200
```

(c) Program Matrix: Energy Flows Arranged by Transformity :

```
      x1  x2  x3  x4
{{1000,-10,0,0},
 {1000,10,-50, 0},
 (1000,10,0,200}};
```

(d) Program Output, Transformity Vector

$$\begin{matrix} x1 \\ x2 \\ x3 \\ x4 \end{matrix} \begin{bmatrix} 1 \\ 100 \\ 40 \\ 10 \end{bmatrix}$$

Figure 4. *Evaluation of co-product pathways with a separate line for each output.*

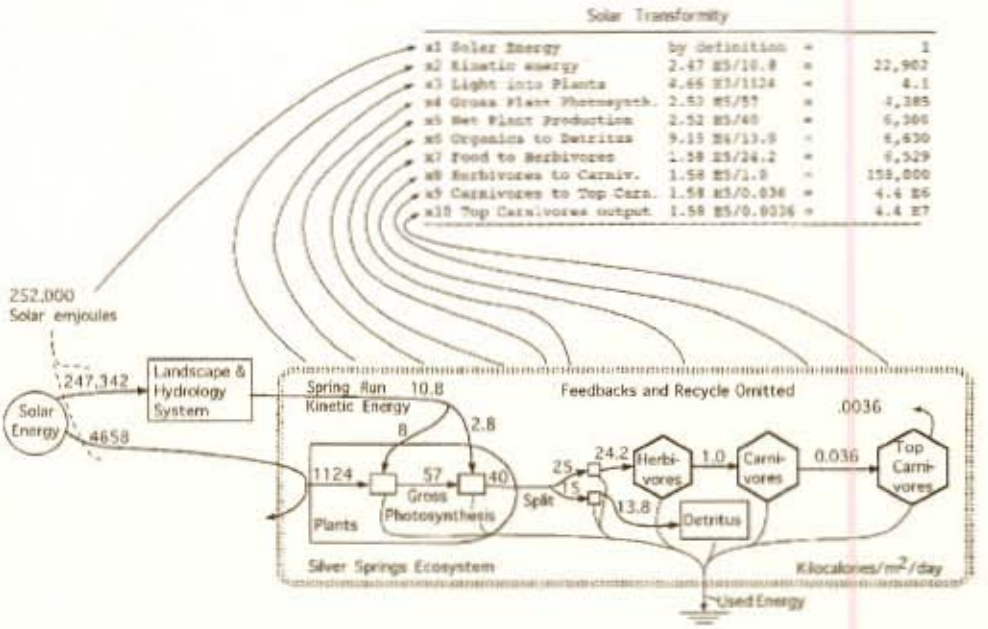


Figure 5. *Energy systems diagram with steady state energy flows for Silver Springs Florida (modified from Odum, 1986). Solar transformities were calculated by dividing energy flows by the contributing emergy. Emergy flow to herbivores and detritus-microzoa is split in proportion to energy branching.*

Evaluating an Ecosystem in Silver Springs, Florida

An evaluation of an ecological system is provided next for Silver Springs, Florida (Odum, 1955), using the energy systems diagram (Odum, 1986). In Figure 5 the ecosystem was aggregated into a long chain, which allows estimation of emergy and transformities by inspection (dividing the input emergy by the output energy at each step). In this procedure closed loops are eliminated since, their net efect on steady state emergy is zero (Odum, 1996, Chapter 6). The emergy equations for the separate energy transformations for Silver Springs are listed as Table 3, and the output of MATHEMATICA program in Table 4 and Table 5. The lowest quality energy is solar insolation, given the value 1, and the values are represented in units of solar transformity (sej/J).

Table 3. Emergy equations for the Silver Springs energy system in Figure 5

x1	Solar energy	247000*x1	= 10.8*x2
x2	kinetic energy	4658*x1	= 1124*x3
x3	Light into plants	8*x2 + 1124*x3	= 57*x4
x4	Gross photosynthesis	2.8*x2 +57*x4	= 40*x5
x5	Net production	15*x5	= 13.8*x6
x6	Organics to detritus	25*x5	= 24.2*x7
x7	Organics to herbivores	24.2*x7	= 1*x8
x8	Herbivores to carnivores	1*x8	= 0.036*x9
x9	Carnivores to top carniv.	0.036*x9	= 0.0036*x10

Table 4. Output of MATHEMATICA evaluating the transformities of Silver Springs energy flows in Figure 5.

```
Out[2]//MatrixForm=
247342  -10.8     0      0      0      0      0      0      0      0
 4658      0   -1124      0      0      0      0      0      0      0
    0      8    1124    -57      0      0      0      0      0      0
    0    2.8       0     57    -40.     0      0      0      0      0
    0      0       0      0     15    -13.8    0      0      0      0
    0      0       0      0     25      0    -24.2    0      0      0
    0      0       0      0      0      0     24.2   -1      0      0
    0      0       0      0      0      0      0      1   -0.036    0
    0      0       0      0      0      0      0      0    0.036  -0.0036

Out[8]= {6.11998×10^10, 2.52796×10^8, 6261.79, 1649.35,
  649.705, 149.381, 14.0594, 0.973349, 0.00130724, 2.16298×10^-17}

Out[11]//MatrixForm=
          1.
       22902.
      4.14413
      3296.04
       6300.
      6847.83
      6508.26
      157500.
    4.375×10^6
    4.375×10^7
```

The energy types in the input matrix headings are the following : x1 = solar energy, x2 = kinetic energy of water, x3 = solar energy to plants, x4 = gross photosynthesis, x5 = net production, x6 = organics to detritus, x7 = energy to herbivores, x8 = energy to carnivores, x9 = energy to top carnivores, x10 = outputs from top carnivores.

Table 5. Comparison of transformities for Silver Springs, Figure 5

	Energy Type	Pathway Ratios*	Mathematica#
x1	Solar energy	1	1
x2	Kinetic energy	2.2 E4	2.3 E4
x3	Light into plants	4.1	4.1
x4	Gross photosynthesis	3.3 E3	4.4 E3
x5	Net production	6.3 E3	6.3 E3
x6	Organics to detritus	6.9 E3	6.6 E3
x7	Organics to herbivores	6.5 E3	6.6 E3
x8	Herbivores to carnivores	1.57 E5	1.58 E5
x9	Carnivores to top carniv.	4.4 E6	4.4 E6
x10	Top carnivores	4.4 E7	4.4 E7

* Quotients from Figure 5
\# Eigenvector from Table 4

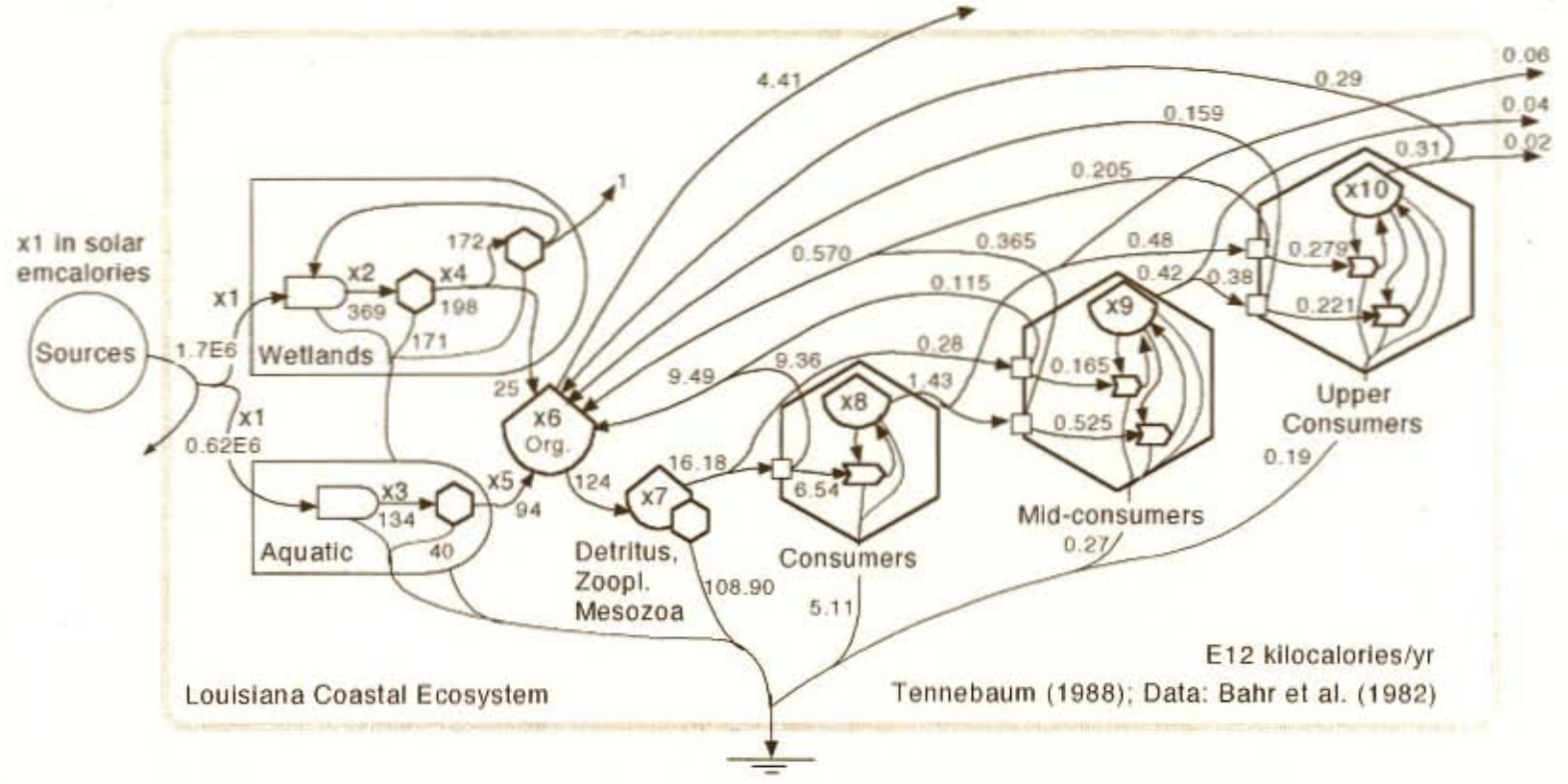


Figure 6. *Energy systems diagram with energy flows for the Louisiana Coastal Ecosystem; modified from Tennenbaum (1988). See Table 6.*

Table 6 . Emergy equations using data from Tennebaum (1988) for energy transformation processes of the Louisiana coastal system in Figure 4 described by Bahr et al. (1982)

Input Energy Flows E12 kcal/yr	Process	Output Flow E12 kcal/yr
x1*1,729,503	Solar wetland use to wetland production	x2*369
x1*628,058	Solar aquatic use to aquatic production	x3*134
x2*369	Wetland prod. to wetland net prod.	x4*198
x3*134	Aquatic prod. to aquat. net prod.	x5*94
x4*25 +x5*94 +x7*9.41 +x8*0.570 +X9*0.159 +x10*0.29	Net production and consumer waste to partic. org.	x6*124
x6*124	Particulate organics to detritus, zooplank., mesozoa	x7*16.18
x7*6.54	Detritus etc. to consumers	x8*1.43
x7*0.165 +x8*0.525	Detritus and consumers to mid consumers	x9*0.42
x8*0.279 + x9*0.221	Consumers and mid consumers to upper consumers	x10*0.31

Evaluating a Louisiana Estuarine System

Tennenbaum (1988) used a pathway tracking method for calculating transformities in an estuarine ecosystem energy network published by Bahr, Day, and Stone (1982). Figure 6 is the energy systems diagram with energy flows on the pathways. The emergy equations for the separate energy transformations are listed as Table 6, and transformity vector from the MATHEMATICA run in Table 7. The network diagram has many splits, some indicated by inter-unit pathways that branch, and others indicated by small squares. The lowest quality energy is solar insolation given the value 1, so that the other values are represented in units of solar transformity (sej/J).

Table 7. Comparison of transformities of the Louisiana coastal ecosystem calculated with three methods.

Item	Track Method[a]	Holistic Inspection[b]	Eigenvector[c]
Solar energy utilized	1	1	1
Gross primary production	4.7 E3	4.7 E3	4.7 E3
Net primary production	8.1 E3	9.3 E3	6.7 E3
Suspended organic matter	1.8 E4	1.8 E4	9.0 E4
Detritus, microbiota, meiofauna	1.36 E5	1.40 E5	6.63 E5
Lower consumers	1.46 E6	1.41 E6	3.04 E6
Mid consumers	4.0 E6	5.6 E6	4.1 E6
Upper consumers	5.2 E6	8.0 E6	5.6 E6

a Pathway track summing by Tennebaum (1988). Explanation given by Odum (1996, page 99).
b Inspection from values on energy systems diagram which are aggregated with few branches.
c Results of MATHEMATICA program (Table 1) with data from Table 6.

COMPARISON OF TRANSFORMITIES WITH DIFFERENT METHODS

In his papers, Patterson uses the term *energy quality equivalents* for the flows of the same energy quality and *quality coefficients* for the values per unit energy. In 1998 he questioned whether these are the same concepts as emergy and transformity. The results from simple evaluation of the energy chain aggregation of the Silver Springs and Louisiana data are similar to those evaluated with the emergy matrix method (Table 5). In Table 7 the results of this computation are compared with those of Tennenbaum using two other methods.

We also evaluated a simplified energy network of New Zealand (Patterson (1993) which he described: "Although this is a hypothetical system, the conversion efficiencies are similar to those that actually occur in the New Zealand energy system." He set electric power as 1 and evaluated the lower quality energy types. When his values were all multiplied by 1.7 E5 solar emjoules per joule electricity, previously established solar transformity for electricity (Odum, 1996), the other kinds of energy had solar transformity values of similar magnitude (in solar emjoules/joule):

	Hypothetical NZ Energy		Odum 1996
Electric Power (set equal)	1.7 E5		1.7 E5
Oil Products	5.4 E4		6.6 E4
Crude Oil	4.4 E4		5.4 E4
Delivered Gas	7.3 E4	Pipeline	4.8 E4

SUMMARY

Eigenvalue-eigenvector computations provide a way to calculate transformities from emergy balance equations, even when data only include a few of the many energy transformation processes in a network. The new procedures in this paper facilitate Murray Patterson's method of computing energy quality relationships from sets of energy transformation relationships. We find that Patterson's terms: "quality equivalent unit" and "quality coefficient" represent the same properties of energy networks as emergy and transformity. It is not true that emergy evaluations are only feed-forward calculations. Emergy evaluations typically include inputs from higher quality (larger scale) as well as from lower quality (smaller scale). This paper provides additional ways for calculating transformity and a convenient program to aid application with commercial software. The program in this paper makes it easy to convert an energy systems diagram that has energy values on its pathways into a set of transformities in a few minutes.

ACKNOWLEDGMENT

We are grateful for the comments and critical review by Murray Patterson and Brian D. Fath.

APPENDIX

A Non-trivial Least Squares Solution of a Homogeneous System
and Proof of the Eigenvalue-Eigenvector method

Dennis Collins

Overview of the method: Some systems analysts have derived systems of homogeneous equations in which the number of processes corresponds to the number m of equations, and the number of commodities corresponds to the number n of variables. The least squares approximation of an m x n system of $Mx = 0$ of homogeneous equations subject to the constraint $x^T x = 1$ is obtained through solving the eigenvalue equation $M^T Mx = \lambda x$. According to the eigenvalue-error formula, the error of selecting a given unit eigenvector as a solution is equal to the eigenvalue corresponding to the given eigenvector, so that the minimum error can be found by taking the eigenvector corresponding to minimum eigenvalue. The least squares method relieves the analyst of the need to match m with n.

First consider that the squared 2-norm error due to selecting a given vector x as the solution of $Mx = 0$ is $x^T M^T Mx$. If this quantity is minimized with the constraint $x^T x = 1$ according to the method of Lagrange multipliers, it is necessary to solve the normal equations $\partial L/\partial Xi = 0$ for $i = 1,2,...,n$ where

$L = x^T M^T Mx - \lambda(x^T x - 1)$ is the Lagrangian function, together with the unit vector constraint.

Calling the symmetric matrix $M^T M = A$, the above equations reduce to

$$2 \sum_{j=1}^{n} A_{ij} x_j - 2 \lambda x_i = 0$$

for $i = 1,2,...,n$ or the eigenvalue equation $Ax = \lambda x$ together with the unit vector constraint. According to "Elementary Linear Algebra" (Prindle Weber Schmidt, 1986) p. 332 by W. Keith Nicholson, the symmetric matrix A has an orthonormal set of eigenvectors.

Next suppose A has two eigenvalues $\lambda 1$ and $\lambda 2$ with $\lambda 1 < \lambda 2$. Let x_1 and x_2 be corresponding eigenvectors, normalized to 1. Then

$$||Mx_1||^2 = x_1^T M^T Mx_1 = x_1^T Ax_1 = x_1^T \lambda_1 x_1 = \lambda_1 < \lambda 2 = x_2^T \lambda_2 x_2$$

$$= x_2^T Ax_2 = ||Mx_2||^2.$$

Stated otherwise (the eigenvalue-error formula), the squared error due to the first eigenvector x_1 is equal to the eigenvalue $\lambda 1$ and less than the squared error due to the second eigenvector x_2. Since the norm is always greater than or equal to 0, the above result also shows that the λ's (defined from $A = M^T M$) are greater than or equal to 0.

Finally, the least squares residue (error) is obtained by taking an eigenvector x with the least eigenvalue (necessarily, greater than or equal to 0). The paper "Working with Projective Space," by the author (1998) explains how to select a unique eigenvector if the eigenspace corresponding to the minimum eigenvalue has dimension greater than one.

In the above equations, it is important to realize that m may be much larger than n, so that the system $Mx = 0$ may have no non-trivial exact solution. Also, the solution does not depend on selection of a "numeraire," but may reflect "quantum learning" if an eigenvector is selected whose eigenvalue is not minimum. In terms of learning theory, the search for x may be considered to be a search for a vector that is nearest to being orthogonal to all the previous rows of M.

If the rows of M are normalized first, say $p = DM$ where D is a diagonal matrix whose ith diagonal entry is $1/d_i$ where d_i is the norm of the ith row of M, then the same analysis as above holds for

the new matrix p (in the terminology of the author's 1998 paper "An approximate Least Squares Method in a Projective-type Space," p. 5), although the eigenvectors of $p^T p$ may be a (and in general will be) different from the eigenvectors of $M^T M$ itself. Thus, the above analysis shows that the exact least-squares error (on the unit sphere) can be obtained by finding the minimum eigenvalue and eigenvector of $p^T p$.

Although the author has still found no reference to the above facts, many others may have discovered them in other settings.

The next two pages work out the exact solution of the Patterson example (cf. "An Approximate Least Squares Method in a Projective-type Space"). The exact least squares error (squared) is found to be 0.029 versus 0.045 for Patterson's method (based on normalized answers); however Patterson's result is still considerably better than the squared error (0.815) of the next-to-least eigenvector, and some part of the extra error may be due to round-off.

Again it must be mentioned that there is a "How to gamble if you must"-aspect about getting the exact minimum residue (error) for the system $Mx = 0$, since it seems more mathematical to normalize the equations (i.e. the rows of M) first and work on the unit sphere, as was done in a previous paper ("An approximate Least Squares Method in a Projective-type space." 1998). However, the above results show the approximation method of that paper gives exact answers for the row-normalized case. Further, the normalization constant (actually the reciprocal) could be considered as a weight factor w_i for the i-th equation, as is sometimes done in least squares theory, thereby bringing the $Mx = 0$ case under the weight-function umbrella.

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